

A HARDY INEQUALITY FOR ULTRASPHERICAL EXPANSIONS WITH AN APPLICATION TO THE SPHERE

ALBERTO ARENAS, ÓSCAR CIAURRI, AND EDGAR LABARGA

ABSTRACT. We prove a Hardy inequality for ultraspherical expansions by using a proper ground state representation. From this result we deduce some uncertainty principles for this kind of expansions. Our result also implies a Hardy inequality on spheres with a potential having a double singularity.

1. INTRODUCTION AND MAIN RESULT

For $d \geq 3$, the classical Hardy inequality states that

$$(1) \quad \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx.$$

Due to its applicability, there is an extensive literature about the topic (see the references in [16]) covering many extensions of this estimate in several and different directions. We are interested in one involving the fractional powers of the Laplacian. We can rewrite (1) as

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^2} dx \leq \int_{\mathbb{R}^d} u(x)(-\Delta u(x)) dx$$

and, taking the fractional Laplacian $(-\Delta)^\sigma$ defined by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma} \widehat{u}$, a natural extension is the inequality

$$(2) \quad C_{\sigma,d} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^{2\sigma}} dx \leq \int_{\mathbb{R}^d} u(x)(-\Delta)^\sigma u(x) dx,$$

for which the sharp constant $C_{\sigma,d}$ is well known (see [3, 20]).

From (2), we deduce the positivity (in a distributional sense) of the operator

$$(-\Delta)^\sigma - \frac{C_{\sigma,d}}{|\cdot|^{2\sigma}}.$$

Our target is to provide a Hardy inequality like (2) related to ultraspherical expansions and apply it to prove the positivity of certain operator on the sphere with a potential having singularities in both poles of the sphere.

Let $C_n^\lambda(x)$ be the ultraspherical polynomial of degree n and order $\lambda > -1/2$. We consider $c_n^\lambda(x) = d_n^{-1} C_n^\lambda(x)$ with

$$d_n^2 = \int_{-1}^1 (C_n^\lambda(x))^2 d\mu_\lambda(x), \quad d\mu_\lambda(x) = (1-x^2)^{\lambda-1/2} dx.$$

2010 *Mathematics Subject Classification.* Primary 42C10.

Key words and phrases. Hardy inequalities, uncertainty principles, ultraspherical expansions.

Research of the second author supported by grant MTM2015-65888-C4-4-P of the Spanish government.

The sequence of polynomials $\{c_n^\lambda\}_{n \geq 0}$ forms an orthonormal basis of the space $L_\lambda^2 := L^2((-1, 1), d\mu_\lambda)$. For each c_n^λ , it holds that $\mathcal{L}_\lambda c_n^\lambda = -(n + \lambda)^2 c_n^\lambda$, where

$$\mathcal{L}_\lambda = (1 - x^2) \frac{d^2}{dx^2} - (2\lambda + 1)x \frac{d}{dx} - \lambda^2.$$

The ultraspherical expansion of each appropriate function f defined in $(-1, 1)$ is given by

$$f \mapsto \sum_{n=0}^{\infty} a_n^\lambda(f) c_n^\lambda,$$

where $a_n^\lambda(f)$ is the n -th Fourier coefficient of f respect to $\{c_n^\lambda\}_{n \geq 0}$, i.e.,

$$a_n^\lambda(f) = \int_{-1}^1 f(y) c_n^\lambda(y) d\mu_\lambda(y).$$

The fractional powers of the operator \mathcal{L}_λ are defined by

$$(-\mathcal{L}_\lambda)^{\sigma/2} f = \sum_{n=0}^{\infty} (n + \lambda)^\sigma a_n^\lambda(f) c_n^\lambda, \quad \sigma > 0.$$

This operator should be the natural candidate to prove a Hardy type inequality for the ultraspherical expansion but, however, it is not the most appropriate in this setting. We have to consider other one with an analogous behaviour to $(-\mathcal{L}_\lambda)^{\sigma/2}$, in order to deduce some results on the sphere. For each $\sigma > 0$ we define (spectrally) the operator

$$A_\sigma^\lambda = \frac{\Gamma(\sqrt{-\mathcal{L}_\lambda} + \frac{1+\sigma}{2})}{\Gamma(\sqrt{-\mathcal{L}_\lambda} + \frac{1-\sigma}{2})}.$$

Then for f defined on the interval $(-1, 1)$

$$A_\sigma^\lambda f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})} a_n^\lambda(f) c_n^\lambda(x).$$

Note that

$$(3) \quad \frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})} \simeq (n + \lambda)^\sigma,$$

then the behaviour of $(-\mathcal{L}_\lambda)^{\sigma/2}$ and A_σ^λ is similar. The natural Sobolev space to analyse Hardy type inequalities is

$$H_\lambda^\sigma = \left\{ f \in L_\lambda^2 : \|f\|_{H_\lambda^\sigma} := \left(\sum_{n=0}^{\infty} (n + \lambda)^\sigma (a_n^\lambda(f))^2 \right)^{1/2} < \infty \right\}.$$

We have to note that H_λ^σ is equivalent to the space $\mathcal{L}_{\lambda,\sigma}^2$ introduced in [5].

With the previous notation our Hardy inequality for ultraspherical expansions is given in the following result.

Theorem 1. *Let $\lambda > 0$ and $0 < \sigma < 1$. Then for $u \in H_\lambda^\sigma$*

$$(4) \quad Q_{\sigma,\lambda} \int_{-1}^1 \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x) \leq \int_{-1}^1 u(x) A_\sigma^\lambda u(x) d\mu_\lambda(x),$$

where

$$(5) \quad Q_{\sigma,\lambda} = 2^\sigma \frac{\Gamma(\frac{\lambda}{2} + \frac{1+\sigma}{4})^2}{\Gamma(\frac{\lambda}{2} + \frac{1-\sigma}{4})^2}.$$

Inequality (4) can be rewritten in terms of the Fourier coefficients

$$(6) \quad Q_{\sigma,\lambda} \int_{-1}^1 \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x) \leq \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2,$$

which is a kind of Pitt inequality for the ultraspherical expansions (for other Pitt inequalities see [4, 11]). Note that for the right hand side of (4) we have, by (3),

$$\int_{-1}^1 u(x) A_\sigma^\lambda u(x) d\mu_\lambda(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 \simeq \|u\|_{H_\lambda^\sigma}^2,$$

so the space H_λ^σ is the adequate one.

The proof of Theorem 1 will be a consequence of a proper ground state representation in our setting, analogous to the given one in the Euclidean case in [9]. Following the ideas in that paper, we can see that the constant $Q_{\sigma,\lambda}$ is sharp but not achieved. Similar ideas have been recently exploited in [7, 16].

From (4), by using Cauchy-Schwarz inequality, we can obtain a Heisenberg type uncertainty principle as it was done for the sublaplacian of the Heisenberg group in [10], and for the fractional powers of the same sublaplacian in [16].

Corollary 2. *Let $\lambda > 0$ and $0 < \sigma < 1$. Then for $u \in H_\lambda^\sigma$*

$$Q_{\sigma,\lambda} \left(\int_{-1}^1 u^2(x) d\mu_\lambda(x) \right)^2 \leq \int_{-1}^1 u^2(x) (1-x^2)^{\sigma/2} d\mu_\lambda(x) \int_{-1}^1 u(x) A_\sigma^\lambda u(x) d\mu_\lambda(x),$$

where $Q_{\sigma,\lambda}$ is the constant given in (5).

Pitt inequality (6) allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions. The main idea comes from [3]. By an elementary argument, for a derivable function such that $\phi(0) = 0$ and $\phi(\sigma) > 0$ for $\sigma \in (0, \varepsilon)$, with $\varepsilon > 0$, it is verified that $\phi'(0_+) \geq 0$. Then, taking the function

$$\phi(\sigma) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 - Q_{\sigma,\lambda} \int_{-1}^1 \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x),$$

we have $\phi(0) = 0$ (this is Parseval identity) and, by (6), $\phi(\sigma) > 0$ for $\sigma \in (0, 1)$, then $\phi'(0_+) \geq 0$ and this inequality gives the logarithmic uncertainty principle, which is written as

$$\begin{aligned} & \left(\log 2 + \psi \left(\frac{\lambda}{2} + \frac{1}{4} \right) \right) \int_{-1}^1 u^2(x) d\mu_\lambda(x) \\ & \leq \sum_{n=0}^{\infty} \psi \left(n + \lambda + \frac{1}{2} \right) (a_n(u))^2 + \int_{-1}^1 \log(\sqrt{1-x^2}) u^2(x) d\mu_\lambda(x), \end{aligned}$$

where $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$.

In next section we will show an application of Theorem 1 to obtain a Hardy inequality on the sphere. The results in Section 3 are the main ingredients in the proof of Theorem 1 which is given in last section of the paper.

2. AN APPLICATION TO THE SPHERE

It is well known that $L^2(\mathbb{S}^d) = \oplus_{n=0}^{\infty} \mathcal{H}_n(\mathbb{S}^d)$, where $\mathcal{H}_n(\mathbb{S}^d)$ is the set of spherical harmonics of degree n in $d+1$ variables. If we consider the shifted Laplacian on the sphere

$$-\Delta_{\mathbb{S}^d} = -\tilde{\Delta}_{\mathbb{S}^d} + \left(\frac{d-1}{2}\right)^2,$$

where $-\tilde{\Delta}_{\mathbb{S}^d}$ is the Laplace-Beltrami operator on \mathbb{S}^d , it is verified that

$$-\Delta_{\mathbb{S}^d} \mathcal{H}_n(\mathbb{S}^d) = \left(n + \frac{d-1}{2}\right)^2 \mathcal{H}_n(\mathbb{S}^d).$$

In this way, the analogous of the operator A_{σ}^{λ} on \mathbb{S}^d is defined by

$$\begin{aligned} \mathbf{A}_{\sigma} f &= \frac{\Gamma(\sqrt{-\Delta_{\mathbb{S}^d}} + \frac{1+\sigma}{2})}{\Gamma(\sqrt{-\Delta_{\mathbb{S}^d}} + \frac{1-\sigma}{2})} f \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{d-1}{2} + \frac{1+\sigma}{2})}{\Gamma(n + \frac{d-1}{2} + \frac{1-\sigma}{2})} \text{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f, \end{aligned}$$

where $\text{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f$ denotes the projection of f onto the eigenspace $\mathcal{H}_n(\mathbb{S}^d)$.

The operator \mathbf{A}_{σ} becomes the fractional powers of the Laplacian in the Euclidean space through conformal transforms as was observed by T. P. Branson in [6]. So \mathbf{A}_{σ} is the natural operator to prove a Hardy type inequality on the sphere. In our proof, we will write \mathbf{A}_{σ} in terms of A_{σ}^{λ} and this is the main reason to consider A_{σ}^{λ} in the case of the ultraspherical expansions. An analogous of the Hardy-Littlewood-Sobolev inequality for \mathbf{A}_{σ} and some other inequalities for it were given by W. Beckner in [2]. The operators \mathbf{A}_{σ} also appear in [18, p. 151] and [17, p. 525].

Each point $x \in \mathbb{S}^d$ can be written as

$$x = (t, \sqrt{1-t^2}x'_1, \dots, \sqrt{1-t^2}x'_d),$$

for $t \in (-1, 1)$ and $x' := (x'_1, \dots, x'_d) \in \mathbb{S}^{d-1}$, and so

$$\int_{\mathbb{S}^d} f(x) dx = \int_{-1}^1 \int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}x') (1-t^2)^{(d-2)/2} dx' dt.$$

With these coordinates, see [19, Section 3], we have that an orthonormal basis for each $\mathcal{H}_n(\mathbb{S}^d)$ is given by

$$\phi_{n,j,k}(x) = \psi_{n,j}(t) Y_{j,k}^d(x'), \quad j = 0, \dots, n,$$

with

$$\psi_{n,j}(t) = (1-t^2)^{j/2} c_{n-j}^{j+(d-1)/2}(t)$$

and $\{Y_{j,k}^d\}_{k=1, \dots, d(j)}$ an orthonormal basis of spherical harmonics on \mathbb{S}^{d-1} of degree j . The value $d(j)$ indicates the dimension of $\mathcal{H}_j(\mathbb{S}^{d-1})$; i.e.,

$$d(j) = (2j+d-2) \frac{(j+d-3)!}{j!(d-2)!}.$$

Then, the orthogonal projection of f onto the eigenspace $\mathcal{H}_n(\mathbb{S}^d)$ can be written as

$$\text{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f = \sum_{j=0}^n \sum_{k=1}^{d(j)} f_{n,j,k} \phi_{n,j,k},$$

with

$$f_{n,j,k} = \int_{-1}^1 G_{j,k}(t) c_{n-j}^{j+(d-1)/2}(t) (1-t^2)^{j+(d-2)/2} dt,$$

$$G_{j,k}(t) = (1-t^2)^{-j/2} F_{j,k}(t) \quad \text{and} \quad F_{j,k}(t) = \int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}x') Y_{j,k}^d(x') dx'.$$

It is easy to observe that

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} F_{j,k}(t) Y_{j,k}^d(x') = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} (1-t^2)^{j/2} G_{j,k}(t) Y_{j,k}^d(x').$$

Moreover, from the definition of \mathbf{A}_σ , we have

$$\mathbf{A}_\sigma f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} (1-t^2)^{j/2} A_\sigma^{j+(d-1)/2} G_{j,k}(t) Y_{j,k}^d(x').$$

Now, considering the Sobolev space

$$\mathbf{H}^\sigma = \left\{ f \in L^2(\mathbb{S}^d) : \|f\|_{\mathbf{H}^\sigma} := \left(\sum_{n=0}^{\infty} \left(n + \frac{d-1}{2} \right)^\sigma \| \text{proj}_{\mathcal{H}_n(\mathbb{S}^d)} f \|_{L^2(\mathbb{S}^d)}^2 \right)^{1/2} < \infty \right\},$$

we have the following Hardy inequality on the sphere.

Theorem 3. *Let $d \geq 2$, $0 < \sigma < 1$, and e_d be the north pole of the sphere \mathbb{S}^d . Then for $f \in \mathbf{H}^\sigma$*

$$(7) \quad 2^\sigma Q_{\sigma,(d-1)/2} \int_{\mathbb{S}^d} \frac{f^2(x)}{(|x - e_d||x + e_d|)^\sigma} dx \leq \int_{\mathbb{S}^d} f(x) \mathbf{A}_\sigma f(x) dx,$$

where $Q_{\sigma,(d-1)/2}$ is the constant given in (5).

Proof. By the orthogonality of the spherical harmonics, it is elementary to show that

$$\int_{\mathbb{S}^d} f(x) \mathbf{A}_\sigma f(x) dx = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \int_{-1}^1 G_{j,k}(t) A_\sigma^{j+(d-1)/2} G_{j,k}(t) d\mu_{j+(d-1)/2}(t).$$

Now, applying Theorem 1, we deduce that

$$\int_{\mathbb{S}^d} f(x) \mathbf{A}_\sigma f(x) dx \geq \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} Q_{\sigma,j+(d-1)/2} \int_{-1}^1 \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}} d\mu_{(d-1)/2}(t).$$

It is known (see [20]) that for $0 < x \leq y$ and $j \geq 0$ we have that $\frac{\Gamma(j+y)}{\Gamma(j+x)} \geq \frac{\Gamma(y)}{\Gamma(x)}$. So, $Q_{\sigma,j+(d-1)/2} \geq Q_{\sigma,(d-1)/2}$ and

$$\int_{\mathbb{S}^d} f(x) \mathbf{A}_\sigma f(x) dx \geq Q_{\sigma,(d-1)/2} \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \int_{-1}^1 \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}} d\mu_{(d-1)/2}(t).$$

The proof of (7) is finished by using the identity

$$\sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \int_{-1}^1 \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}} d\mu_{(d-1)/2}(t) = 2^\sigma \int_{\mathbb{S}^d} \frac{f^2(x)}{(|x - e_d||x + e_d|)^\sigma} dx.$$

□

The analogous role on the sphere of radially symmetric functions is played by functions which are invariant under the action of $SO(d-1)$. By $SO(d-1)$ -invariance we mean that f is invariant under the action of the group $SO(d-1)$ on \mathbb{S}^{d-1} whenever $SO(d-1)$ is embedded into $SO(d)$ in a suitable way. Each function f of this kind can be written as $f(x) = g(\langle x, e_d \rangle)$, for a certain function g defined in $(-1, 1)$. Then for this kind of functions Theorem 3 reduces to Theorem 1 with $\lambda = (d-1)/2$, in this way we can deduce that the constant $2^\sigma Q_{\sigma, (d-1)/2}$ in (7) is sharp.

As in the classic case, from Theorem 3 we deduce that in a distributional sense

$$\mathbf{A}_\sigma - \frac{2^\sigma Q_{\sigma, (d-1)/2}}{(|x - e_d||x + e_d|)^\sigma} \geq 0.$$

Note that in this case we are perturbing the operator \mathbf{A}_σ adding a potential with singularities in both poles of the sphere.

3. AUXILIARY RESULTS

The following lemmas give the tools to prove Theorem 1. To be more precise, Lemma 1 provides a nonlocal representation of the operator A_σ^λ with a kernel having nice properties for our target. Lemma 2 shows the action of the operator A_σ^λ on the family of weights $(1 - x^2)^{-(\lambda/2 + (1-\sigma)/4)}$.

For $f, g \in L_\lambda^2$ we are going to set up the notation

$$\langle f, g \rangle_\lambda = \int_{-1}^1 f(x)g(x) d\mu_\lambda(x)$$

to simplify the writing.

Lemma 1. *Let $\lambda > 0$ and $0 < \sigma < 1$. If f is a finite linear combination of ultraspherical polynomials, then*

$$(8) \quad A_\sigma^\lambda f(x) = \int_{-1}^1 (f(x) - f(y)) K_\sigma^\lambda(x, y) d\mu_\lambda(y) + E_{\sigma, \lambda} f(x), \quad x \in (-1, 1),$$

where the kernel is given by

$$K_\sigma^\lambda(x, y) = D_{\sigma, \lambda} \int_{-1}^1 \frac{d\mu_{\lambda-1/2}(t)}{(1 - xy - \sqrt{1-x^2}\sqrt{1-y^2}t)^{\lambda+(1+\sigma)/2}},$$

with

$$D_{\sigma, \lambda} = \frac{c_\lambda^2}{2^{\lambda+(1+\sigma)/2}} \frac{\Gamma(\frac{1-\sigma}{2})\Gamma(\lambda + \frac{1+\sigma}{2})}{|\Gamma(-\sigma)|\Gamma(1+\lambda)}, \quad c_\lambda = \frac{\Gamma(2\lambda+1)}{2^{2\lambda}(\Gamma(\lambda+1/2))^2},$$

and

$$E_{\sigma, \lambda} = \frac{\Gamma(\lambda + \frac{1+\sigma}{2})}{\Gamma(\lambda + \frac{1-\sigma}{2})}.$$

Moreover, for $f \in H_\lambda^\sigma$ we have

$$(9) \quad \langle A_\sigma^\lambda f, f \rangle_\lambda = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (f(x) - f(y))^2 K_\sigma^\lambda(x, y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma, \lambda} \langle f, f \rangle_\lambda$$

Proof. We start with the identity

$$(10) \quad \int_0^\infty \left(e^{-(n+\lambda)t} - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} dt = 2^{1+\sigma} \Gamma(-\sigma) \frac{\Gamma(n+\lambda + \frac{1+\sigma}{2})}{\Gamma(n+\lambda + \frac{1-\sigma}{2})}$$

for $\lambda > 0$ (actually it is also true for values $\lambda > -1/2$) and $0 < \sigma < 1$. To deduce the previous identity it is enough to apply integration by parts with $u = e^{-(n+\lambda+(1-\sigma)/2)t} - 1$ and $v = -2e^{-\sigma t/2}(\sinh t/2)^{-\sigma}/\sigma$, and use [14, eq. 8, p. 367]

$$\int_0^\infty e^{-\rho t} (\cosh(ct) - 1)^\nu dt = \frac{\Gamma(\frac{\rho}{c} - \nu)\Gamma(2\nu + 1)}{2^\nu c \Gamma(\frac{\rho}{c} + \nu + 1)}$$

for $c > 0$, $2\nu > -1$, and $\rho > c\nu$.

Now, we consider the Poisson operator for ultraspherical expansions. It is given by

$$e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) = \sum_{n=0}^\infty e^{-(n+\lambda)t} a_n^\lambda(f) c_n^\lambda(x) = \int_{-1}^1 f(y) P_t^\lambda(x, y) d\mu_\lambda(y),$$

with

$$P_t^\lambda(x, y) = \sum_{n=0}^\infty e^{-(n+\lambda)t} c_n^\lambda(x) c_n^\lambda(y).$$

By the product formula for ultraspherical polynomials [8, eq. B.2.9, p. 419]

$$\frac{C_n^\lambda(x) C_n^\lambda(y)}{C_n^\lambda(1)} = c_\lambda \int_{-1}^1 C_n^\lambda(xy + \sqrt{1-x^2}\sqrt{1-y^2}t) d\mu_{\lambda-1/2}(t), \quad \lambda > 0,$$

the identity [8, eq. B.2.8. p. 419]

$$\sum_{n=0}^\infty \frac{n+\lambda}{\lambda} C_n^\lambda(x) r^n = \frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}}, \quad 0 \leq r < 1,$$

and the relation $d_n^2 = \frac{\lambda}{c_\lambda(n+\lambda)} C_n^\lambda(1)$, we deduce the expression

$$P_t^\lambda(x, y) = \frac{c_\lambda^2}{2^\lambda} \int_{-1}^1 \frac{\sinh t}{(\cosh t - w(s))^{\lambda+1}} d\mu_{\lambda-1/2}(s),$$

with $w(s) = xy + \sqrt{1-x^2}\sqrt{1-y^2}s$. The previous identity for P_t^λ is not new, it appears as formula (2.12) in [12].

Combining (10) and the definition of the Poisson operator, it is clear that

$$A_\sigma^\lambda f(x) = \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x) e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} dt,$$

which can be splitted in

$$\begin{aligned} (11) \quad A_\sigma^\lambda f(x) &= \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x) e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) \right) (\sinh t/2)^{-\sigma-1} dt \\ &\quad + \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} dt. \end{aligned}$$

From the obvious identity

$$e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) = \int_{-1}^1 P_t^\lambda(x, y) d\mu_\lambda(y) = e^{-\lambda t},$$

for the second term in (11) we have

$$\begin{aligned} & \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} dt \\ &= \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-\lambda t} - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} dt \\ &= E_{\sigma,\lambda} f(x), \end{aligned}$$

where we have used (10) with $n = 0$.

The first integral in (11) verifies

$$\begin{aligned} & \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\mathcal{L}_\lambda}} f(x) - f(x) e^{-t\sqrt{-\mathcal{L}_\lambda}} 1(x) \right) (\sinh t/2)^{-\sigma-1} dt \\ &= \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_0^\infty \int_{-1}^1 P_t^\lambda(x, y) (f(x) - f(y)) d\mu_\lambda(y) (\sinh t/2)^{-\sigma-1} dt \\ &= \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_{-1}^1 (f(x) - f(y)) \int_0^\infty P_t^\lambda(x, y) (\sinh t/2)^{-\sigma-1} dt d\mu_\lambda(y) \\ &= \int_{-1}^1 (f(x) - f(y)) K_\sigma^\lambda(x, y) d\mu_\lambda(y), \end{aligned}$$

with

$$K_\sigma^\lambda(x, y) = \frac{1}{2^{1+\sigma}|\Gamma(-\sigma)|} \int_0^\infty P_t^\lambda(x, y) (\sinh t/2)^{-\sigma-1} dt.$$

In last computation we have used Fubini theorem. This is justified for finite combinations of ultraspherical polynomials by using the estimate

$$P_t^\lambda(x, y) \leq \frac{C \sinh t}{(1-x^2)^{\lambda/2} (1-y^2)^{\lambda/2} (\cosh t - xy - \sqrt{1-x^2} \sqrt{1-y^2})},$$

which follows from the elementary inequality

$$\int_{-1}^1 \frac{(1-s^2)^{\lambda-1}}{(A-Bs)^{\lambda+1}} ds \leq \frac{C}{B^\lambda(A-B)}, \quad A > B > 0, \quad \lambda > 0,$$

and the mean value theorem. Indeed, taking $C_f = \max\{|f'(x)| : x \in [-1, 1]\}$ and using the inequality $1 - xy - \sqrt{1-x^2} \sqrt{1-y^2} \geq C|x-y|^2$, we have

$$\begin{aligned} & \int_0^\infty \int_{-1}^1 P_t^\lambda(x, y) |f(x) - f(y)| d\mu_\lambda(y) (\sinh t/2)^{-\sigma-1} dt \\ & \leq \frac{C_f}{(1-x^2)^{\lambda/2}} \left(C_1 \int_0^\infty \int_{-1}^1 \frac{t^{-\sigma} |x-y|}{t^2 + |x-y|^2} (1-y^2)^{\lambda/2-1/2} dy dt \right. \\ & \quad \left. + C_2 \int_1^\infty \int_{-1}^1 e^{-(\sigma+1)t/2} |x-y| (1-y^2)^{\lambda/2-1/2} dy dt \right) =: \frac{C_f}{(1-x^2)^{\lambda/2}} (I_1 + I_2). \end{aligned}$$

Obviously, I_2 is a finite integral. For I_1 the change of variable $t = |x-y|s$ gives

$$I_1 \leq C_1 \int_0^\infty \frac{s^{-\sigma}}{s^2 + 1} ds \int_{-1}^1 |x-y|^{-\sigma} (1-y^2)^{\lambda/2-1/2} dy < \infty.$$

To obtain the expression of K_σ^λ we observe that

$$\begin{aligned} K_\sigma^\lambda(x, y) &= \frac{c_\lambda^2}{2^{\lambda+1+\sigma} |\Gamma(-\sigma)|} \int_0^\infty \int_{-1}^1 \frac{\sinh t}{(\cosh t - w(s))^{\lambda+1}} d\mu_{\lambda-1/2}(s) (\sinh t/2)^{-\sigma-1} dt \\ &= \frac{c_\lambda^2}{2^{\lambda+(1+\sigma)/2}} \frac{\Gamma(\frac{1-\sigma}{2})\Gamma(\lambda + \frac{1+\sigma}{2})}{|\Gamma(-\sigma)|\Gamma(\lambda+1)} \int_{-1}^1 \frac{d\mu_{\lambda-1/2}(s)}{(1-w(s))^{\lambda+(1+\sigma)/2}}, \end{aligned}$$

where we have applied Fubini theorem and the change of variable $2(\sinh t/2)^2 = z(1-w(s))$ in last equality. With the last identity we have concluded the proof of (8).

To prove (9) we follow the argument in [16, Lemma 5.1]. First, we observe that the kernel $K_\sigma^\lambda(x, y)$ is positive and symmetric in the sense that $K_\sigma^\lambda(x, y) = K_\sigma^\lambda(y, x)$. Then, (9) is clear when f is a finite linear combination of ultraspherical polynomials. For $f \in H_\sigma^\lambda$ we consider a sequence of finite linear combinations of ultraspherical polynomials $\{p_k\}_{k \geq 0}$ such that p_k converges to f in H_σ^λ . Then, by using the definition of A_σ^λ , it is clear that $\langle A_\sigma^\lambda p_k, p_k \rangle_\lambda$ converges to $\langle A_\sigma^\lambda f, f \rangle_\lambda$. Moreover, the result for polynomial functions implies

$$(12) \quad \langle A_\sigma^\lambda p_k, p_k \rangle_\lambda = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (p_k(x) - p_k(y))^2 K_\sigma^\lambda(x, y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma, \lambda} \langle p_k, p_k \rangle_\lambda < \infty.$$

Consequently, the functions $P_k(x, y) = p_k(x) - p_k(y)$ form a Cauchy sequence in $L^2((-1, 1) \times (-1, 1), d\omega)$ where $d\omega(x, y) = K_\sigma^\lambda(x, y) d\mu_\lambda(x) d\mu_\lambda(y)$ which converges to $f(x) - f(y)$ in this norm. Hence, passing to the limit in (12), we complete the proof of the lemma. \square

Lemma 2. *Let $\lambda > 0$ and $2\lambda + 1 > \sigma > 0$. Then*

$$(13) \quad A_\sigma^\lambda \left(\frac{1}{(1-x^2)^{\lambda/2+(1-\sigma)/4}} \right) = \frac{Q_{\sigma, \lambda}}{(1-x^2)^{\lambda/2+(1+\sigma)/4}},$$

where $Q_{\sigma, \lambda}$ is the constant given in (5).

Proof. First of all, we have to realize that the ultraspherical polynomial $C_n^\lambda(x)$ is odd for $n = 2m+1$, $m \in \mathbb{Z}^+$; therefore, for $\beta > 0$, the function $(1-x^2)^{\beta-1} C_{2m+1}^\lambda(x)$ is an odd function and its integral over the interval $(-1, 1)$ is zero. For $n = 2m$ we use [15, eq. 15, p. 519] to obtain

$$\begin{aligned} & \int_{-1}^1 (1-x^2)^{\beta-1} C_{2m}^\lambda(x) dx \\ &= \sqrt{\pi} \frac{(2\lambda)_{2m}}{(2m)!} \frac{\Gamma(\beta)}{\Gamma(\beta+1/2)} {}_3F_2(-2m, 2\lambda+2m, \beta; 2\beta, \lambda+1/2; 1) \\ &= \pi \frac{(2\lambda)_{2m}}{(2m)!} \frac{\Gamma(\beta)\Gamma(\lambda+1/2)\Gamma(\beta-\lambda+1/2)}{\Gamma(1/2-m)\Gamma(\lambda+m+1/2)\Gamma(\beta+m+1/2)\Gamma(\beta-\lambda-m+1/2)}, \end{aligned}$$

where in last identity we have evaluated the hypergeometric function with the so-called Watson formula [13, eq. 16.4.6, p. 406]. Therefore, if we denote $\alpha = \lambda/2 + (1-\sigma)/4$, we obtain that

$$(14) \quad \int_{-1}^1 (1-x^2)^{\alpha-1} C_{2m}^\lambda(x) dx = R_{\sigma, \lambda} \int_{-1}^1 (1-x^2)^{\alpha+\sigma/2-1} C_{2m}^\lambda(x) dx,$$

with

$$R_{\sigma,\lambda} = \frac{\Gamma(\alpha)\Gamma(\alpha - \lambda + 1/2)}{\Gamma(\alpha + \sigma/2)\Gamma(\alpha - \lambda + 1/2 + \sigma/2)} \times \frac{\Gamma(\alpha + m + 1/2 + \sigma/2)\Gamma(\alpha - \lambda - m + 1/2 + \sigma/2)}{\Gamma(\alpha + m + 1/2)\Gamma(\alpha - \lambda - m + 1/2)}.$$

In this way, if we prove the identity

$$(15) \quad R_{\sigma,\lambda} = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(2m + 2\alpha + \sigma)}{\Gamma(2m + 2\alpha)}$$

we will conclude the proof, because (14) implies

$$a_n^\lambda \left(\frac{1}{(1-x^2)^{\alpha+\sigma/2}} \right) = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(n+2\alpha+\sigma)}{\Gamma(n+2\alpha)} a_n^\lambda \left(\frac{1}{(1-x^2)^\alpha} \right),$$

where we have had in mind that the n -th Fourier coefficient is null when $n = 2m+1$.

Let us check that (15) actually holds. Using the reflection formula [1, eq. 6.1.17, p. 256] twice we have

$$\begin{aligned} \frac{\Gamma(\alpha - \lambda - m + 1/2 + \sigma/2)}{\Gamma(\alpha - \lambda - m + 1/2)} &= \frac{\Gamma(\alpha + m + \sigma/2)}{\Gamma(\alpha + m)} \frac{\sin(\pi(\alpha - \lambda - m + 1/2))}{\sin(\pi(\alpha - \lambda - m + 1/2 + \sigma/2))} \\ &= \frac{\Gamma(\alpha + m + \sigma/2)}{\Gamma(\alpha + m)} \frac{\Gamma(\alpha)\Gamma(\alpha - \lambda + 1/2 + \sigma/2)}{\Gamma(\alpha + \sigma/2)\Gamma(\alpha - \lambda + 1/2)}, \end{aligned}$$

and then

$$\begin{aligned} R_{\sigma,\lambda} &= \frac{\Gamma(\alpha)^2}{\Gamma(\alpha + \sigma/2)^2} \frac{\Gamma(\alpha + m + \sigma/2)\Gamma(\alpha + m + \sigma/2 + 1/2)}{\Gamma(\alpha + m)\Gamma(\alpha + m + 1/2)} \\ &= Q_{\sigma,\lambda}^{-1} \frac{\Gamma(2m + 2\alpha + \sigma)}{\Gamma(2m + 2\alpha)}, \end{aligned}$$

by the duplication formula [1, eq. 6.1.18, p. 256]. \square

4. PROOF OF THEOREM 1

Polarizing the identity (9) in Lemma 1 we obtain

$$(16) \quad \langle g, A_\sigma^\lambda f \rangle_\lambda = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 F(x, y) K_\sigma^\lambda(x, y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma,\lambda} \langle g, f \rangle_\lambda,$$

with $F(x, y) = (g(x) - g(y))(f(x) - f(y))$.

Let us take $g(x) = (1 - x^2)^{-\lambda/2 - (1-\sigma)/4}$ and $f(x) = u^2(x)/g(x)$ for $u \in H_\lambda^\sigma$. Then

$$F(x, y) = (u(x) - u(y))^2 - g(x)g(y) \left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2$$

and (16) becomes

$$\begin{aligned} &\langle g, A_\sigma^\lambda f \rangle_\lambda \\ &= \langle u, A_\sigma^\lambda u \rangle_\lambda - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 g(x)g(y) \left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2 K_\sigma^\lambda(x, y) d\mu_\lambda(y) d\mu_\lambda(x). \end{aligned}$$

Now, by (13), we have

$$\langle g, A_\sigma^\lambda f \rangle_\lambda = \langle A_\sigma^\lambda g, f \rangle_\lambda = Q_{\sigma,\lambda} \int_{-1}^1 \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x)$$

and then we can deduce the ground state representation

$$\begin{aligned}
 (17) \quad & \langle u, A_\sigma^\lambda u \rangle_\lambda - Q_{\sigma, \lambda} \int_{-1}^1 \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x) \\
 &= \frac{1}{2} \int_{-1}^1 \int_{-1}^1 g(x)g(y) \left(\frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2 K_\sigma^\lambda(x, y) d\mu_\lambda(y) d\mu_\lambda(x).
 \end{aligned}$$

So, due to the positivity of the kernel K_σ^λ , we conclude the proof.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, 55, Washington, 1964.
- [2] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality, *Ann. of Math. (2)* **138** (1993), 213–242.
- [3] W. Beckner, Pitt’s inequality and the uncertainty principle, *Proc. Amer. Math. Soc.* **123** (1995), 1897–1905.
- [4] W. Beckner, Pitt’s inequality with sharp convolution estimates, *Proc. Amer. Math. Soc.* **136** (2008), 1871–1885.
- [5] J. J. Betancor, J. C. Faria, L. Rodríguez-Mesa, R. Testoni, and J. L. Torrea, L. A choice of Sobolev spaces associated with ultraspherical expansions, *Publ. Mat.* **54** (2010), 221–242.
- [6] T. P. Branson, Sharp inequalities, the functional determinant, and the complementary series, *Trans. Amer. Math. Soc.* **347** (1995), 3671–3742.
- [7] Ó. Ciaurri, L. Roncal, and S. Thangavelu, Hardy–type inequalities for fractional powers of the Dunkl–Hermite operator, *preprint*, arXiv:1602.04997.
- [8] F. Dai and Y. Xu, *Approximation theory and Harmonic Analysis on spheres and balls*, Springer, New York, 2013.
- [9] R. L. Frank, E. H. Lieb, and R. Seiringer, Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators, *J. Amer. Math. Soc.* **21** (2008), 925–950.
- [10] N. Garofalo and E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier (Grenoble)* **40** (1990), 313–356.
- [11] D. V. Gorbachev, V. I. Ivanov, and S. Yu Tikhonov, Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L^2 , *J. Approx. Theory* **202** (2016), 109–118.
- [12] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17–92.
- [13] F. W. J. Olver (editor-in-chief), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.
- [14] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*. Vol. 1. Elementary functions. Translated from the Russian and with a preface by N. M. Queen. Gordon and Breach Science Publishers, New York, 1986.
- [15] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*. Vol. 2. Special functions. Translated from the Russian by N. M. Queen. Gordon and Breach Science Publishers, New York, 1986.
- [16] L. Roncal and S. Thangavelu, Hardy’s inequality for fractional powers of the sublaplacian on the Heisenberg group, *Adv. Math.* **302** (2016), 106–158.
- [17] B. Rubin, *Introduction to Radon transforms. With elements of fractional calculus and harmonic analysis*, Encyclopedia of Mathematics and its Applications, 160, Cambridge University Press, New York, 2015.
- [18] S. G. Samko, *Hypersingular integrals and their applications*, Analytical Methods and Special Functions, 5, Taylor & Francis, London, 2002.
- [19] T. O. Sherman, The Helgason Fourier transform for compact Riemannian symmetric spaces of rank one, *Acta Math.* **164** (1990), 73–144.
- [20] D. Yafaev, Sharp constants in the Hardy–Rellich inequalities, *J. Funct. Anal.* **168** (1999), 121–144.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, COMPLEJO
CIENTÍFICO-TECNOLÓGICO, CALLE MADRE DE DIOS 53, 26006, LOGROÑO, SPAIN

E-mail address: `alarenas@unirioja.es`

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, COMPLEJO
CIENTÍFICO-TECNOLÓGICO, CALLE MADRE DE DIOS 53, 26006, LOGROÑO, SPAIN

E-mail address: `oscar.ciaurri@unirioja.es`

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, COMPLEJO
CIENTÍFICO-TECNOLÓGICO, CALLE MADRE DE DIOS 53, 26006, LOGROÑO, SPAIN

E-mail address: `edlabarg@unirioja.es`